

Dynamics of Dusty Pair-Ion-Electron Plasma Modeled by the Cylindrical Kadomtsev-Petviashvili Equations

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Abstract: The nonlinear propagation and stability of dust ion-waves in plasma is analytically and numerically investigated. By using the standard reductive perturbation method, the electrostatic potential in dusty pair-ion-electron plasma is modelled by cylindrical Kadomtsev-Petviashvili (CKP) equation. The soliton solutions are obtained using the direct integration for single soliton solution and the Hirota bilinear method to find multisoliton solution of the system. It is noticed that the Hirota method better illustrate the physical reality of dust pair-ion plasma since it generalizes different forms of solutions. From the numerical simulations, it is observed that, the plasma parameters strongly influence the properties of the soliton solution, namely, the amplitude and the width. The analysis of the stability of the soliton solutions reveals that the stable solution co-propagates with seven other solutions, eigenmodes of the Legendre equation. These modes contain basic symmetry and axisymmetric configuration consistent with relevant experimental observations in existing experiments.

Keywords: CKP Equation, Periodic Soliton Trains, Stability, Internal Modes, Propagation Modes

1. Introduction

The electron-ion Plasma type exhibit a variety of collective modes due to large mass difference between the electron and the ion; whereas in pair-ion plasma, the masses are equal and the charges are opposite [1, 2]. This pair-ion-plasma has been widely studied for the fundamental understanding of environment such as ionosphere, magnetosphere, quasars, etc [1-17]. Examples of such plasma are: Hydrogen pair-ion plasma produced by catalytic ionization, electron-positron plasma produced by introducing electron in positron plasma, pair-ions plasma of fullerenes C_{60}^{\pm} experimentally produced in laboratory through the impact ionization of fullerenes [3]. On like other pair-ion plasma, the advantage of fullerene C_{60}^{\pm} plasma lies in its stability and greater annihilation time than the plasma period, making it very easy to understand the collective mode of such plasma [1, 3, 4]. The simplicity in the theoretical description of pair-ion plasma makes it very interesting for investigations. Recently, some authors demonstrated the existence of electrons in the fullerene pair-ion plasma as a result of the experimental observation of

acoustic waves within such plasma [1, 3, 4].

In the presence of dust particle such as dust acoustic (DA), dust ion acoustic (DIA) or dust lattice (DL), the collective modes of pair-ion-electron plasma can be affected [5]. A such plasma, called Dusty plasma (or complex, colloidal, and aerosol plasma) [6], is an ordinary plasma contaminated with a certain amount of condensed (solid or liquid) particles (grains). The study of dusty plasmas represents one of the most rapidly growing field of plasma physics [7-11]. Their theoretical features and applications observed in Earth magnetosphere, ionosphere, mesosphere, cometary tail, and planetary rings make them very attractive [12-16]. Various types of dusty pair-ion plasmas can be found both in space and in laboratory. Experimental observations have confirmed the linear and non-linear features of both dust acoustic waves (DAW) and dust ion acoustic waves (DIAW) [12, 17].

Note that, in most investigations, reductive perturbation method has been used for deriving the Korteweg-de Vries (KdV) equation in the Cartesian coordinate system with one dimension. This is sometimes insufficient to describe the physical reality in laboratory and space plasmas. Studies

have also been carried on the dynamic of dusty plasma in the two-dimensional Cartesian coordinates in order to get closer to reality [18]. This revealed that the amplitude of propagating waves is greater than those gotten in a one-dimensional coordinate [15]. The experimental conditions mentioned in many laboratories show that cylindrical coordinates must be imposed in current work to move from the purely mathematical environment to a more realistic physical environment [19]. However, Wei-Qi Peng *et al* carried out studies on pair-ion electron plasma in cylindrical coordinates, where they were able to analytically demonstrate the propagation of breather and rogue waves soliton [3]. Also, in reference [4], a one dimensional Cartesian coordinate study of dusty pair-ion-electron plasma was carried out and revealed the influence of some plasma parameter on the soliton wave. All these studies do not give a clear picture of the real physical situation. Linear and nonlinear plasma properties, containing massive charged dust particles, can be significantly affected. In particular, they can result in the appearance of new normal modes, such as the dust-acoustic and dustion-acoustic waves [20].

In this study, we applied for the first time a polar coordinate system to a particular dusty pair-ion electron plasma model to derive the cylindrical Kadomtsev-Petviashvili equation (CKP). We use direct integration and Hirota analytical techniques to solve this equation. Various soliton solutions describing the dynamics in dusty pair-ion electron plasmas are obtained. Note that, wave phenomena are important for heating plasmas, instabilities and diagnostics. The presence of dust grains significantly modifies the linear and nonlinear wave propagation through the plasma.

The rest of the paper is organized as follows: in section II, we present the model and derive the CKP equation using the reductive perturbation method (RPM). Section III, illustrates the single soliton solution using direct integration; and the single and multi-soliton solutions by Hirota's bilinear method. In section IV, we will analyze the stability of the solutions obtained in section III. The work end with a conclusion in section V.

2. Mathematical Model

The fundamental wave modes in a dusty plasma is constituted of an unmagnetized plasma made up of two principal modes namely the Dust ion acoustic (DIA) mode and the Dust acoustic wave (DAW). Magnetized plasma also have two principal modes known as the Electrostatic dust ion cyclotron wave (EDIC) and the Electrostatic dust cyclotron wave (EDC) [12]. To study the role of dust particle in the pair-ion-electron, we consider this dust particle in an unmagnetized pair-ion-electron plasma which generates the dust acoustic mode. In this model we consider the dust particles to be dynamic, but we ignore the gravitational effect and the inertia of the electrons and ions due to their masses [21]. We assume also that the positive, negative ions and electrons follow the Maxwell Boltzmann distribution

functions given by

$$\begin{cases} n_+ = n_{+0} e^{-e\phi/T_+} \\ n_- = n_{-0} e^{-e\phi/T_-} \\ n_e = n_{e0} e^{-e\phi/T_e}, \end{cases} \quad (1)$$

where $n_j (j = +, -, e)$ are perturbed densities of positive ion, negative ion and electron respectively [1]. In cylindrical geometry, the dust particle in dusty plasma is governed by the following normalize equations

$$\frac{\partial n_d}{\partial t} + \nabla(n_d v_d) = 0 \quad (2)$$

$$\frac{\partial v_d}{\partial t} + (v_d \cdot \nabla) v_d = s \frac{e Z_d}{m_d} E \quad (3)$$

$$\nabla \cdot E = 4\pi e (n_+ - n_- - n_e + s Z_d n_d), \quad (4)$$

In which eqns.(2), (3), and (4) are the continuity, momentum, and Poisson equations respectively. In equilibrium state, there is no electrical field. Hence the total charge is neutral at equilibrium [14] and the Poisson equation becomes:

$$n_{+0} - n_{-0} - n_{e0} + s Z_d n_{d0} = 0 \quad (5)$$

The subscript d_0 denotes the dust particle at equilibrium, $s = \pm 1$ is use to indicate the positive or negative dust particle, E characterizes the electric field related to the electrostatic potential Φ by $E = -\nabla\Phi$, Z_d and v_d are the charge number and velocity of dust particles respectively; and n_+ , n_- and n_e represent the densities of the ions and electrons. The equilibrium densities of these quantities are n_{+0} , n_{-0} and n_{e0} . We introduce normalized variables such that the perturbed velocity components (v_r, v_θ) are normalized by the ion acoustic speed of the dust particles $C_{sd} = \sqrt{T_+/m_d}$, where T_+ is the temperature of the positive ions. The perturbed densities for the ions, electrons and dust particles are normalized with their equilibrium densities; and the normalized electrostatic potential becomes $\Phi = e\phi/T_+$. The space and time variables are normalized through the Debye length $\lambda_D = \sqrt{T_+ / (4\pi n_{d0} Z_d^2 e^2)}$ and plasma frequency $\omega_{pd} = \sqrt{(4\pi n_{d0} Z_d^2 e^2) / m_d}$ respectively. We also define the following ratios $\beta = T_+ / T_-$, $\delta = n_{-0} / s Z_d n_{d0}$, $\sigma = T_+ / T_e$, and $\mu = n_{+0} / s Z_d n_{d0}$, where δ and μ are related by $\mu = \delta - 1 + n_{e0} / s Z_d n_{d0}$. This lead to the he normalized forms of equations (1) to (4) expressed in cylindrical geometry as:

$$\frac{\partial n_d}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r n_d v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (n_d v_\theta) = 0 \quad (6)$$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = -s Z_d \frac{\partial \Phi}{\partial r}, \quad (7)$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{v_r v_\theta}{r} = -\frac{sZ_d}{r} \frac{\partial \Phi}{\partial \theta}, \quad (8)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = \frac{s}{Z_d} [(\delta - \mu - 1)n_e + \delta n_- - \mu n_+ - n_d], \quad (9)$$

$$n_+ = e^{-\Phi}; n_- = e^{\beta\Phi}; n_e = e^{\sigma\Phi}. \quad (10)$$

To reduce equations (6)-(10), we make use of the standard reductive perturbation method and re-scaling both space and time in the equations, the variables are labelled in power series of ε as:

$$\begin{cases} n_d = 1 + \varepsilon n_{d1} + \varepsilon^2 n_{d2} + \varepsilon^3 n_{d3} + \dots \\ n_- = 1 + \varepsilon n_{-1} + \varepsilon^2 n_{-2} + \varepsilon^3 n_{-3} + \dots \\ n_+ = 1 + \varepsilon n_{+1} + \varepsilon^2 n_{+2} + \varepsilon^3 n_{+3} + \dots \\ n_e = 1 + \varepsilon n_{e1} + \varepsilon^2 n_{e2} + \varepsilon^3 n_{e3} + \dots \\ v_r = \varepsilon v_{r1} + \varepsilon^2 v_{r2} + \varepsilon^3 v_{r3} + \dots \\ v_\theta = \varepsilon^{3/2} v_{\theta1} + \varepsilon^{5/2} v_{\theta2} + \varepsilon^{7/2} v_{\theta3} + \dots \\ \Phi = \varepsilon \Phi_1 + \varepsilon^2 \Phi_2 + \varepsilon^3 \Phi_3 + \dots \end{cases} \quad (11)$$

Where ε ($0 < \varepsilon < 1$) is a small expansion parameter proportional to the amplitude [1, 4, 6, 12], The Taylor series expansions of the densities are:

$$\begin{cases} n_+ = 1 - \Phi + \frac{1}{2} \Phi^2 + \dots \\ n_- = 1 + \beta\Phi + \frac{1}{2} \beta^2 \Phi^2 + \dots \\ n_e = 1 + \sigma\Phi + \frac{1}{2} \sigma^2 \Phi^2 + \dots \end{cases} \quad (12)$$

The scaled independent variables can be define as:

$$\begin{cases} X = \varepsilon^{1/2} (r - \lambda t), \\ Y = \varepsilon^{-1/2} \theta \\ \tau = \varepsilon^{3/2} t, \end{cases} \quad (13)$$

where λ represents the normalize wave velocity. Making use of equation (13) we obtain the following equations:

$$\begin{cases} r = \frac{X}{\varepsilon^{1/2}} + \lambda t, \\ \frac{\partial}{\partial t} = \varepsilon^{3/2} \frac{\partial}{\partial \tau} - \varepsilon^{1/2} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial r} = \varepsilon^{1/2} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial \theta} = \varepsilon^{-1/2} \frac{\partial}{\partial Y}. \end{cases} \quad (14)$$

Substituting equations (10)-(14) in equations (6)-(9), we obtain equations of various power of ε . For the lowest order terms of ε , the continuity and the momentum equations are expressed as:

$$n_{d1} = \frac{v_{r1}}{\lambda}, \quad v_{r1} = \frac{sZ_d}{\lambda} \Phi_1, \quad \frac{\partial v_{\theta1}}{\partial r} = \frac{sZ_d}{\lambda^2 \tau} \frac{\partial \Phi_1}{\partial \eta}, \quad (15)$$

while the Poisson equation yields:

$$(\delta - \mu - 1)n_{e1} + \delta n_{-1} - \mu n_{+1} - n_{d1} = 0. \quad (16)$$

Hence, the Boltzmann distribution of ions and electrons can be deduced as follows:

$$\begin{cases} n_{e1} = \sigma \Phi_1 \\ n_{-1} = \beta \Phi_1 \\ n_{+1} = \Phi_1 \\ n_{d1} = \frac{sZ_d}{\lambda^2} \Phi_1. \end{cases} \quad (17)$$

Introducing equation (17) into equation (16), we derive the following dispersion relation of the electrostatic dust wave of pair-ion-electron plasma as

$$(\delta - \mu - 1)\sigma + \delta\beta + \mu - \frac{sZ_d}{\lambda^2} = 0 \quad (18)$$

For the next higher order terms of both the continuity and the momentum equations, we obtain:

$$\begin{aligned} \lambda \tau \frac{\partial n_{d1}}{\partial \tau} - \lambda^2 \tau \frac{\partial n_{d2}}{\partial X} + 2\lambda^2 \tau n_{d1} \frac{\partial n_{d1}}{\partial X} + \lambda \tau \frac{\partial v_{r2}}{\partial X} + \\ \frac{\partial v_{\theta1}}{\partial Y} + v_{r1} = 0, \end{aligned} \quad (19)$$

$$\begin{aligned} -\lambda^2 X \frac{\partial n_{d1}}{\partial X} + \lambda^2 \tau \frac{\partial n_{d1}}{\partial \tau} - \lambda^2 \tau \frac{\partial v_{r2}}{\partial X} + \\ \lambda^3 \tau n_{d1} \frac{\partial n_{d1}}{\partial X} = -sZ_d \left(X \frac{\partial \Phi_1}{\partial X} - \lambda \tau \frac{\partial \Phi_2}{\partial X} \right), \end{aligned} \quad (20)$$

while for the Poisson equation, we have:

$$\frac{\partial^2 \Phi_1}{\partial X^2} = \frac{s}{Z_d} [(\delta - \mu - 1)n_{e2} + \delta n_{-2} - \mu n_{+2} - n_{d2}], \quad (21)$$

Where we define $n_{+2} = -\Phi_2 + \frac{1}{2} \Phi_1^2$; $n_{-2} = \beta\Phi_2 + \frac{1}{2} \beta^2 \Phi_1^2$

and $n_{e2} = \sigma\Phi_2 + \frac{1}{2} \sigma^2 \Phi_1^2$.

Next, we rewrite equation (21) as

$$\frac{\partial^3 \Phi_1}{\partial X^3} = \frac{s}{Z_d} \left(\kappa \Phi_1 \frac{\partial \Phi_1}{\partial X} + \frac{sZ_d}{\lambda^2} \frac{\partial \Phi_2}{\partial X} - \frac{\partial n_{d2}}{\partial X} \right), \quad (22)$$

Where $\kappa = \sigma^2(\delta - \mu - 1) + \beta^2\delta - \mu$

After some mathematical manipulations, we derive the following equation in term of Φ_1

$$\frac{\partial}{\partial X} \left(\frac{\partial \Phi_1}{\partial \tau} + A\Phi_1 \frac{\partial \Phi_1}{\partial X} + B \frac{\partial^3 \Phi_1}{\partial X^3} + \frac{\Phi_1}{2\tau} \right) + \frac{C}{\tau^2} \frac{\partial^2 \Phi_1}{\partial Y^2} = 0, \quad (23)$$

which denotes the cylindrical Kadomtsev-Petviashvili (CPK) equation, with

$$A = \left(\frac{3sZ_d}{2\lambda} - \frac{\kappa\lambda^3}{2sZ_d} \right), \quad B = \frac{\lambda^2}{2s^2}, \quad C = \frac{1}{2\lambda}, \quad (24)$$

A , B and C characterize the non linear coefficient, the dispersion coefficient and the weak dispersion coefficient respectively. It is important to remark that, if we neglect the Y dependence, the CKP equation (23) is reduce to the one dimension KDV equation [4].

Making use of the following mapping $X' = X - \frac{Y^2\tau}{4C}$,

$$Y' = Y\tau \quad \text{and} \quad \tau' = \tau, \quad \text{with} \quad \frac{\partial}{\partial X} = \frac{\partial X'}{\partial X} \frac{\partial}{\partial X'},$$

$$\frac{\partial}{\partial Y} = \frac{\partial X'}{\partial Y} \frac{\partial}{\partial X'} + \frac{\partial Y'}{\partial Y} \frac{\partial}{\partial Y'}, \quad \frac{\partial}{\partial \tau} = \frac{\partial X'}{\partial \tau} \frac{\partial}{\partial X'} + \frac{\partial Y'}{\partial \tau} \frac{\partial}{\partial Y'} + \frac{\partial \tau'}{\partial \tau} \frac{\partial}{\partial \tau'},$$

The CPK equation (23) can be reduced to an ordinary Kadomtsev-Petviashvili (KP) equation given

$$\frac{\partial}{\partial X'} \left(\frac{\partial \Phi_1}{\partial \tau'} + A\Phi_1 \frac{\partial \Phi_1}{\partial X'} + B \frac{\partial^3 \Phi_1}{\partial X'^3} \right) + \frac{C}{\tau'^2} \frac{\partial^2 \Phi_1}{\partial Y'^2} = 0 \quad (25)$$

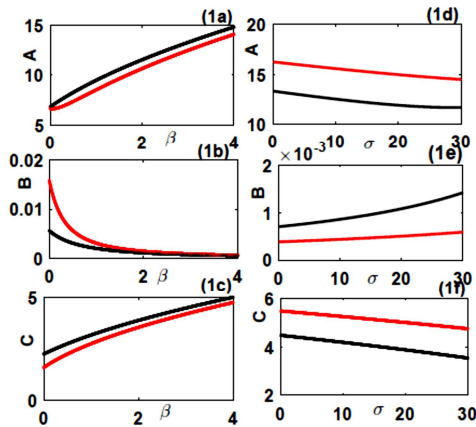


Figure 1. Representation of the non linear coefficient A , the dispersion coefficient B and the weak dispersion coefficient C profiles in function of β for two different values of σ ; $\sigma = 0.001$ (black line) and $\sigma = 10$ (red line) (Figure 1a-1c) and in function of σ for two different values of β ; $\beta = 3$ (black line) and $\beta = 5$ (red line) (Figure 1e-1f). The others parameters used are: $a=b=1$; $u=8$; $s=1$; $Z_d=1$; $n_{d0}=0.05n_{-0}$; $n_{e0}=0.05n_{-0}$; $\delta = \frac{n_{-0}}{n_{d0}sZ_d}$

$$\text{and } \mu = \delta - 1 + \frac{n_{e0}}{n_{d0}sZ_d}.$$

Figure 1, shows the profiles of the coefficients A , B and C as function of the dusty plasma parameters β and σ . Figure 1a

and 1c indicate that A and C increase with respect to β , while in figure 1b, B decreases as β increases for two different fixed values of σ . This physically point out that an increase in temperature of positive ions relative to negative ions, strengthen the nonlinearity (characterized by A) and weak dispersion (characterized by C) at the detriment of the dispersion coefficient (i.e. B). Figure 1e presents an increase in B while in figure 1d and 1f a decrease of A and C with respect to σ for two fixed values of β is observed. That is, the nonlinearity and weak dispersion are reduced for a decreasing temperature of electrons. Hence, with a suitable choice of the dusty plasma parameters we can have a perfect compensation between non linearity and dispersions leading to solitary wave's propagation in the milieu.

3. The CPK Equation Solutions

We aim at this point to find solitary wave solutions of the CKP equation. For this reason, by making choice of a suitable change of frame with speed u , equation (25) can be transform to following form:

$$a \frac{\partial}{\partial \xi} \left(-u \frac{\partial \Phi_1}{\partial \xi} + Aa^2\Phi_1 \frac{\partial \Phi_1}{\partial \xi} + Ba^4 \frac{\partial^3 \Phi_1}{\partial \xi^3} \right) + Cb^2 \frac{\partial^2 \Phi_1}{\partial \xi^2} = 0, \quad (26)$$

Where $\xi = aX' + bY' - u\tau'$

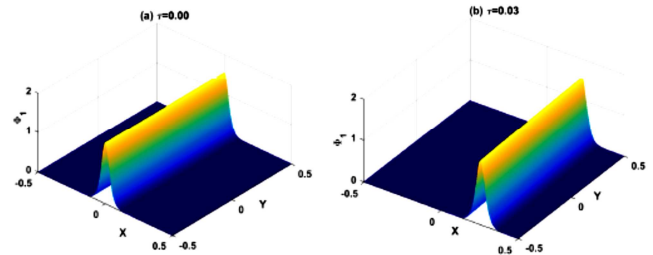


Figure 2. Three-dimensional representation of the electrostatic potential profile in cylindrical geometry at two different times $\tau = 0$ and $\tau = 0.03$: The parameters used are: $a=b=1$; $u=8$; $s=1$; $Z_d=1$; $\beta=1$; $\sigma=0.01$; $n_{d0}=0.05n_{-0}$; $\delta = \frac{n_{-0}}{n_{d0}sZ_d}$ and $\mu = \delta - 1 + \frac{n_{e0}}{n_{d0}sZ_d}$.

3.1. The Single Soliton Solution by Direct Integration

From equation (25), we carry out two consecutive integrations under the conditions that

$$\Phi_1 = 0, \quad \frac{\partial \Phi_1}{\partial \xi} = 0, \quad \frac{\partial^2 \Phi_1}{\partial \xi^2} = 0 \quad \text{for } \Phi \rightarrow \infty, \quad \text{equation (26)}$$

can be transformed as:

$$(Cb^2 - au)\Phi_1 + \frac{Aa^2}{2}\Phi_1^2 + Ba^4 \frac{\partial^2 \Phi_1}{\partial \xi^2} = 0 \quad (27)$$

By integrating equation (27), we obtain a hyperbolic secant pulse in the form:

$$\Phi_1 = \Phi_m \text{sech}^2\left(\frac{\xi}{W}\right) \quad (28)$$

which is a single soliton wave solution of the KP, with the amplitude parameter $\Phi_m = \frac{3(au - Cb^2)}{a^2 A}$, and the width parameter $W = \sqrt{\frac{4a^4 B}{au - Cb^2}}$. In a 3D representation, the solitary wave solution of CKP equation in function of X, Y and τ is given as

$$\Phi_1 = \frac{3(au - Cb^2)}{a^2 A} \times \text{sech}^2 \left[\sqrt{\frac{au - Cb^2}{4a^4 B}} \left[a \left(X - \frac{Y^2 \tau}{4C} \right) + bY\tau - u\tau \right] \right] \quad (29)$$

Figure 2 represents the time evolution of the 3D solitary wave solution in cylindrical geometry at two different time $\tau = 0$ and $\tau = 0.03$; We precise that when the value of τ increases, the amplitude and the width of the soliton remain unchanged as indicated in figure 2.

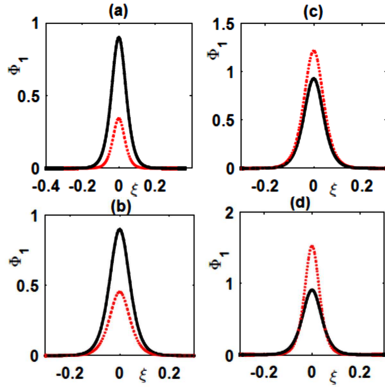


Figure 3. Two dimensional representation of the electrostatic potential profile in cylindrical geometry for two different values of β : $\beta = 1$ (Solid line) and $\beta = 3$ (Dashed line) (Figure 3a) for two different values of σ : $\sigma = 0.1$ (Solid line) and $\sigma = 0.8$ (Dashed line) (Figure 3b), for two different values of Z_d : $Z_d = 1$ (Solid line) and $Z_d = 2$ (Dashed line) (Figure 3c); for two different values of u : $u = 6$ (Solid line) and $u = 8$ (Dashed line) (Figure 3d). The others parameters used are: $a = b = 1$; $u = 6$ and $u = 8$; $s = 1$; $Z_d = 1$ or $Z_d = -1$; $\sigma = 0.1$ or $\sigma = 0.5$; $n_{d0} = 0.05n_{-0}$; $n_{e0} = 0.05n_{-0}$; $\delta = \frac{n_{-0}}{n_{d0}sZ_d}$ and $\mu = \delta - 1 + \frac{n_{e0}}{n_{d0}sZ_d}$.

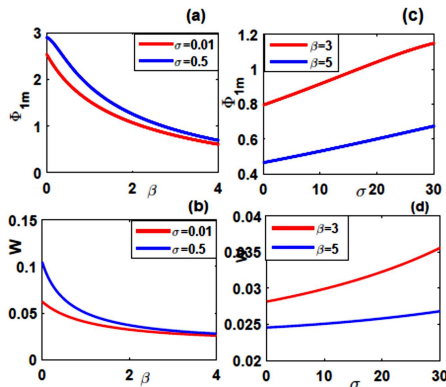


Figure 4. Variation of amplitude and width of solution in function of β for two different values of σ : $\sigma = 0.01$ (red line) and $\sigma = 0.5$ (blue line) (Figures 1a and 1b) and in function of σ for two different values of β : $\beta = 3$ (red line) and $\beta = 5$ (blue line) (Figures 1c and 1d). The parameters used are: $a = b = 1$; $u = 8$; $s = 1$; $Z_d = 1$; $n_{d0} = 0.05n_{-0}$; $n_{e0} = 0.05n_{-0}$; $\delta = \frac{n_{-0}}{n_{d0}sZ_d}$; $\mu = \delta - 1 + \frac{n_{e0}}{n_{d0}sZ_d}$.

The figures 3 and 4 shows the effects of plasma parameters on the amplitude of the electrostatic potential Φ_m and the width (or spatial extension) W of the localized pulse respectively. In particular, the figure illustrate that the amplitude and width decrease as β (i.e. the ratio of temperature between the positive and negative ions). On the contrary, Φ_m and W increase with σ (i.e. ratio in temperature between the positive ions and electrons). Thus, the presences of charged dusts significantly affect the properties of solitons.

3.2. Single and Multiple Solution Using Hirota Bilinear Form

To understand complex nonlinear phenomena in dusty plasma, many researchers focused on various types of soliton solution and their remarkable properties in many branches of physics including plasma physics. Among the analytical methods used in the resolution of nonlinear differential equations, we can name the bilinear method which consists in putting a differential equation in an integrable form called the bilinear form of Hirota [22-30]. Hirota bilinear method is a powerful tool for obtaining a wide class of exact solutions of some nonlinear equations. It is one of the most famous methods to construct multi-soliton solutions. The idea of this method is to construct a dependent variable transformation called Hirota transformation, also called the tau-function transformation, which transforms a nonlinear equation into a bilinear equation known as a Hirota bilinear equation [28]. The Hirota bilinear operator D is define as follow [22-27]:

$$D_x(f.g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) f(x)g(x')_{x=x'}, \quad D_x^m D_t^n = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x)g(x')_{x=x', t=t'}. \quad (30)$$

The compact form of the Hirota derivative is given by:

$$D_x^m f.g = \sum (-1)^r C_m^r f_{(m-r)x} g_{rx}, \quad (31)$$

where, f_x stand for the derivative of f with respect to x , and

$$C_m^r = \frac{m!}{r!(m-r)!} [24, 25, 30].$$

The Hirota method introduces the auxillary function F which is the logarithmic transformation of Φ_1 . This transformation is constructed as follows [3, 19, 27]

$$\Phi_1 = \frac{12B}{A} (\ln F)_{XX'} = \frac{12B}{A} \left(\frac{FF_{XX'} - F_{X'}^2}{F^2} \right). \quad (32)$$

Replacing equation (32) in equation (25), we obtain the following expression:

$$FF_{X'\tau} - F_{X'}F_\tau + B(F_{XX'XX'}F - 4F_{XX'X'}F_{X'} + 3F_{XX'}F_{X'X'}) + C(FF_{Y\tau} - F_Y^2) = 0. \quad (33)$$

Then, making use of the definition of bilinear operator, we obtain the bilinear form of the KP equation given below:

$$(D_{X'}D_\tau + BD_{X'}^4 + CD_Y^2)F.F = 0, \quad (34)$$

which is in the form $P(D)(F.F) = 0$ with

$P(D) = (D_X' D_{\tau'} + B D_X'^4 + C D_Y'^2)$ being a polynomial in the Hirota partial derivatives D . We impose the function F to take the following form $F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \dots$, so that the multisoliton solutions can be obtained by finite perturbation around the vacuum ($F = 1$) [32]. The above scheme is generally applicable to any explicit bilinear operator expression. It can be shown that if the original partial differential equation admits N -soliton solution, then equation (34) will be truncated as $n = N$ terms, provided F is the sum of precisely N simple exponential terms [25, 30]. Hence, this leads to:

3.2.1. One Soliton Solution

$F = 1 + \varepsilon f_1$ and equation (34) is written as

$$P(D)[(1 + \varepsilon f_1).(1 + \varepsilon f_1)] = 0 \quad (35)$$

After some simplifications we obtain:

$$P(D)[1.1 + (1.f_1 + f_1.1)\varepsilon + f_1.f_1\varepsilon^2] = 0 \quad (36)$$

We can then deduce the following equation:

$$\frac{\partial f_1}{\partial X' \partial \tau'} + B \frac{\partial f_1^4}{\partial X'^4} + C \frac{\partial f_1^2}{\partial Y'^2} = 0. \quad (37)$$

The solution of equation (37) can be found in the form [29-30]:

$$f_1 = e^\eta, \quad (38)$$

With η in the form $\eta = pX' + qY' - \Omega\tau'$. Replacing equation (38) in equation (37), we obtain the dispersion relation

$$\Omega = \left[Bp^3 + C \frac{q^2}{p} \right] \quad (39)$$

Thus, we obtain the following solution

$$F = 1 + e^{(pX' + qY' - \Omega\tau')}, \quad (40)$$

where we took $\varepsilon = 1$ without loss generality. Finally, the solution of CKP equation in cylindrical coordinate with variables X , Y , and τ as

$$\Phi_1 = \frac{3Bp^2}{A} \times \text{sech}^2 \left[\frac{1}{2} \left[p \left(X - \frac{Y^2\tau}{4C} \right) + qY\tau - \left(Bp^3 + C \frac{q^2}{p} \right) \tau \right] \right] \quad (41)$$

The profile of this solution is identical to the one obtained in the case of the single soliton solution when $N = 1$, which is a particular case of multi soliton solution. The figure 5 depict the time evolution of the one soliton solution. We observe that, as the value of τ increases, the amplitude of wave remain unchange while the structure of the soliton changes from a line shape to a parabolic shape; many authors obtain the same result in others plasma models, the one dimensional

geometry model cannot show such behavior [1, 19].

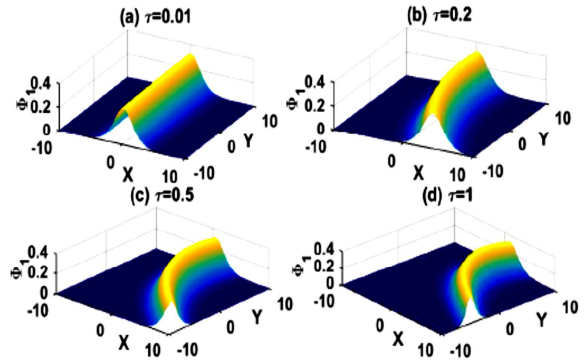


Figure 5. Three-dimensional representation of one soliton solution profile obtain from Hirota bilinear method at four different times $\tau = 0.01(a)$; $\tau = 0.2(b)$; $\tau = 0.5(c)$ and $\tau = 1(d)$: The parameter used are: $p = q = 1$, $s=1$; $Z_d = 1$; $\beta = 1$; $\sigma = 0.01$; $n_{d0} = 0.05n_{-0}$; $n_{e0} = 0.05n_{-0}$, $\delta = \frac{n_{-0}}{n_{d0}sZ_d}$ and $\mu = \delta - 1 + \frac{n_{e0}}{n_{d0}sZ_d}$.

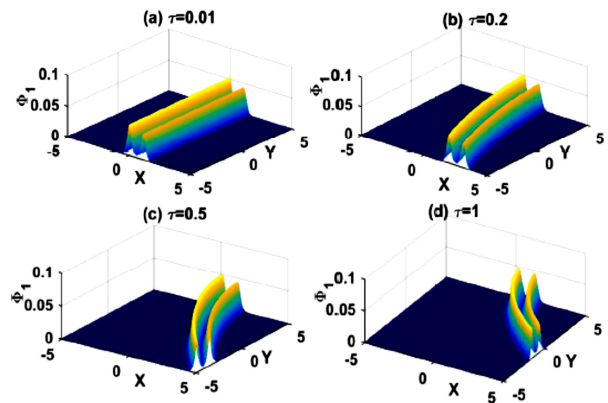


Figure 6. Three-dimensional representation of two solitons profile obtain from Hirota bilinear method at four different times $\tau = 0.01(a)$; $\tau = 0.2(b)$; $\tau = 0.5(c)$ and $\tau = 1(d)$: The parameters used are: $p_1 = q_1 = 8$, $p_2 = q_2 = 10$; $s=1$; $Z_d = 1$; $\beta = 1$; $\sigma = 0.01$; $n_{d0} = 0.05n_{-0}$; $n_{e0} = 0.05n_{-0}$, $\delta = \frac{n_{-0}}{n_{d0}sZ_d}$ and $\mu = \delta - 1 + \frac{n_{e0}}{n_{d0}sZ_d}$.

3.2.2. Two-Soliton Solutions

$F = 1 + f_1\varepsilon + f_2\varepsilon^2$ and the KP equation take the form

$$P(D)(1 + f_1\varepsilon + f_2\varepsilon^2)(1 + f_1\varepsilon + f_2\varepsilon^2) = 0 \quad (42)$$

By making used properties of Hirota operators we can obtain the following equations

$$P(D)(1.1)=0; \quad P(D)(1.f_1+f_1.1)=0; \quad P(D)(1.f_2+f_1.f_1+f_2.1)=0 \quad (43)$$

Since we seek for two-soliton solution, we use the twoterm form of f_i with the ansatz f_2 given respectively as:

$$f_1 = e^{\eta_1} + e^{\eta_2} \text{ and } f_2 = a_{12}e^{\eta_1+\eta_2} \quad (44)$$

where $\eta_1 = p_1X' + q_1Y' - \Omega_1\tau'$ and $\eta_2 = p_2X' + q_2Y' - \Omega_2\tau'$.

a_{12} is a coupling constant given by:

$$a_{12} = -\frac{P(\eta_1 - \eta_2)}{P(\eta_1 + \eta_2)} = -\frac{(\Omega_1 - \Omega_2)(p_1 - p_2) + B(p_1 - p_2)^4 + C(q_1 - q_2)^2}{(\Omega_1 + \Omega_2)(p_1 + p_2) + B(p_1 + p_2)^4 + C(q_1 + q_2)^2}. \quad (45)$$

with $\eta_i = (p_i; q_i; \Omega_i)$ [26].

In the context of our dusty plasma model the following dispersion relations are obtained as

$$\Omega_1 = \left[Bp_1^3 + C \frac{q_1^2}{p_1} \right], \quad \Omega_2 = \left[Bp_2^3 + C \frac{q_2^2}{p_2} \right] \quad (46)$$

Using the similar process in the case of one-soliton solution, i.e. $F = 1 + f_1 + f_2 = 1 + e^{\eta_1} + e^{\eta_2} + a_{12}e^{\eta_1 + \eta_2}$, where we have taken $\varepsilon = 1$, the CPK two-soliton solution is finally obtain as

$$\Phi_1 = \frac{12B}{A} (\ln F)_{X'X'} = \frac{12B}{A} \left(\frac{FF_{X'X'} - F_{X'}^2}{F^2} \right). \quad (47)$$

Where F , $F_{X'}$ and $F_{X'X'}$ are expressed as:

$$\begin{aligned} F &= 1 + e^{\eta_1} + e^{\eta_2} + a_{12}e^{\eta_1 + \eta_2}, \\ F_{X'} &= p_1e^{\eta_1} + p_2e^{\eta_2} + a_{12}(p_1 + p_2)e^{\eta_1 + \eta_2} \text{ and} \\ F_{X'X'} &= p_1^2e^{\eta_1} + p_2^2e^{\eta_2} + a_{12}(p_1 + p_2)^2e^{\eta_1 + \eta_2}. \end{aligned} \quad (48)$$

With $X' = X - \frac{Y^2\tau}{4C}$, $Y' = Y\tau$, and $\tau' = \tau$.

Figure 6 presents the three dimensional representation of

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + a_{12}e^{\eta_1 + \eta_2} + a_{13}e^{\eta_1 + \eta_3} + a_{23}e^{\eta_2 + \eta_3} + a_{123}e^{\eta_1 + \eta_2 + \eta_3}, \quad (54)$$

Where, again without loss of generality, we have set $\varepsilon = 1$. As in the previous cases, we replace F , $F_{X'}$, $F_{X'X'}$, X' , Y' and τ' in the equation (32) to obtain three-soliton solution of CKP equation in this case,

$$\begin{aligned} F_{X'} &= p_1e^{\eta_1} + p_2e^{\eta_2} + p_3e^{\eta_3} + a_{12}(p_1 + p_2)e^{\eta_1 + \eta_2} + a_{13}(p_1 + p_3)e^{\eta_1 + \eta_3} + a_{23}(p_2 + p_3)e^{\eta_2 + \eta_3} + a_{123}(p_1 + p_2 + p_3)e^{\eta_1 + \eta_2 + \eta_3} \\ F_{X'X'} &= p_1^2e^{\eta_1} + p_2^2e^{\eta_2} + p_3^2e^{\eta_3} + a_{12}(p_1 + p_2)^2e^{\eta_1 + \eta_2} + a_{13}(p_1 + p_3)^2e^{\eta_1 + \eta_3} + a_{23}(p_2 + p_3)^2e^{\eta_2 + \eta_3} + a_{123}(p_1 + p_2 + p_3)^2e^{\eta_1 + \eta_2 + \eta_3} \end{aligned} \quad (55)$$

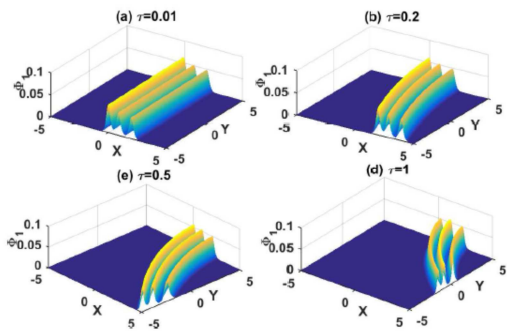


Figure 7. Two and three-dimensional representations of three solitons profile obtain from Hirota bilinear method at four different times $\tau = 0.01$ (a); $\tau = 0.2$ (b); $\tau = 0.5$ (c) and $\tau = 1$ (d). The parameters used are: $p_1 = q_1 = 8$, $p_2 = q_2 = 10$; $p_3 = q_3 = 12$; $s = 1$; $Z_d = 1$; $\beta = 1$; $\sigma = 0.01$; $n_{d0} = 0.05n_{-0}$; $n_{e0} = 0.05n_{-0}$; $\delta = \frac{n_{-0}}{n_{d0}sZ_d}$ and $\mu = \delta - 1 + \frac{n_{e0}}{n_{d0}sZ_d}$.

the two-soliton solutions profiles in cylindrical coordinate propagating in the plasma model under consideration. like in one soliton case, the structure becomes a parabolic shape as the time increases..

3.2.3. Three-Solitons Solutions

For the three-soliton solution, we chose:

$$F = 1 + f_1\varepsilon + f_2\varepsilon^2 + f_3\varepsilon^3, \quad (49)$$

With $f_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}$, $f_2 = a_{12}e^{\eta_1 + \eta_2} + a_{13}e^{\eta_1 + \eta_3} + a_{23}e^{\eta_2 + \eta_3}$ and $f_3 = a_{123}e^{\eta_1 + \eta_2 + \eta_3}$, where, $\eta_1 = p_1X' + q_1Y' - \Omega_1\tau'$, $\eta_2 = p_2X' + q_2Y' - \Omega_2\tau'$ and $\eta_3 = p_3X' + q_3Y' - \Omega_3\tau'$

The coupling constant in this case are given as

$$a_{12} = -\frac{P(\eta_1 - \eta_2)}{P(\eta_1 + \eta_2)} = -\frac{(\Omega_1 - \Omega_2)(p_1 - p_2) + B(p_1 - p_2)^4 + C(q_1 - q_2)^2}{(\Omega_1 + \Omega_2)(p_1 + p_2) + B(p_1 + p_2)^4 + C(q_1 + q_2)^2}, \quad (50)$$

$$a_{13} = -\frac{P(\eta_1 - \eta_3)}{P(\eta_1 + \eta_3)} = -\frac{(\Omega_1 - \Omega_3)(p_1 - p_3) + B(p_1 - p_3)^4 + C(q_1 - q_3)^2}{(\Omega_1 + \Omega_3)(p_1 + p_3) + B(p_1 + p_3)^4 + C(q_1 + q_3)^2}, \quad (51)$$

$$a_{23} = -\frac{P(\eta_2 - \eta_3)}{P(\eta_2 + \eta_3)} = -\frac{(\Omega_2 - \Omega_3)(p_2 - p_3) + B(p_2 - p_3)^4 + C(q_2 - q_3)^2}{(\Omega_2 + \Omega_3)(p_2 + p_3) + B(p_2 + p_3)^4 + C(q_2 + q_3)^2}, \quad (52)$$

$$\text{And } a_{123} = a_{12}a_{13}a_{23} \quad (53)$$

The parameters p_i , q_i and Ω_i are required to satisfy the dispersion relation $\Omega_i = \left[Bp_i^3 + C \frac{q_i^2}{p_i} \right]$. The auxiliary function F give:

Figure 7 illustrate the graphical profile representation in 3D of the three soliton solution. It is worth noting that as the number of pulses increases, the amplitudes decrease, leading to the complete vanishing of the pulse. The behavior of three soliton structure with the time is identical to previous case.

Thus, for any $N > 3$, the N -soliton solutions can always be constructed in a similar way. However, the calculations become very lengthy.

4. Stability Analysis of Solutions

To analyze the stability of the solution, we introduce a small perturbation in the solution and study his evolution. Stability analysis is important for the understanding of the robustness of the solution when propagating [31]. In order to study the stability of the solution, we perturb the initial solution Φ_i of equation (26) and define a perturbed solution as:

$$\Phi_1(\xi) = \Phi_{10}(\xi) + \epsilon \Phi_{11}(\xi), \quad (56)$$

where ϵ is a small parameter that separates the soliton solution $\Phi_{10}(\xi)$ and the perturbation profile $\Phi_{11}(\xi)$. We introduce equation (56) into the reduce form of equation (27) to obtain the following equations at consecutive order of ϵ .

At order ϵ^0 , we get

$$L\Phi_{10} + S\Phi_{10}^2 + T\frac{\partial^2\Phi_{10}}{\partial\xi^2} = 0, \quad (57)$$

at order ϵ^1 ,

$$L\Phi_{11} + 2S\Phi_{10}\Phi_{11} + T\frac{\partial^2\Phi_{11}}{\partial\xi^2} = 0. \quad (58)$$

The soliton solution of equation (57) is in the form given by equation (28):

$$\Phi_{10} = \Phi_m \text{sech}^2\left(\frac{\xi}{W}\right) \quad (59)$$

Considering the reduced form of equation (27), this solution can be rewritten as:

$$\Phi_1 = -\frac{3L}{2S} \text{sech}^2\left(\sqrt{-\frac{L}{4T}}\xi\right), \text{ Where } L = Cb^2 - au, \quad S = \frac{Aa^2}{2}$$

and $T = Ba^4$.

By introducing this solution into equation (58) leads us to:

$$\frac{\partial^2\Phi_{11}}{\partial\xi^2} + \omega^2[-4 + 12\text{sech}^2(\omega\xi)]\Phi_{11} = 0, \quad (60)$$

Where $\omega = \sqrt{-\frac{L}{4T}}$. By setting $Y = \omega\xi$, we have:

$$\frac{\partial^2\Phi_{11}}{\partial Y^2} + [-4 + 12\text{sech}^2(Y)]\Phi_{11} = 0 \quad (61)$$

Equation (61) represents the associate Legendre equation which solutions are eigen-vectors corresponding to Legendre Polynomials [32]. Thus the solution will be stable if and only if equation (61) is fulfilled. Recall that, the Legendre equation which has been widely studied is an eigenvalue problem and it possesses both discrete and continuous modes, all describing background modes of the localized solution Φ_1 . However here we are mainly interested in fields with a localized shape profile. In this respect we consider only boundstate modes of the eigenvalue equation (61), which correspond to wave functions of the discrete branch [32, 33]. Here, we aim at finding solution $\Phi_{11}(Y)$ of equation (61), in which case the solution of the system will be stable. This background wave has seven distinct localized modes $\Phi_{11}^j(k)(j=1,2,3,4,5,6,7)$ (see Appendix). Figure 8 below shows the variation of theses solutions who represent other profile that can be obtained in our plasma model. The propagation of the electrostatic potential is possible only in one of these modes, and is mainly

specified by the physical conditions. Following this, the behaviour of dusty Pair-Ion-Electron is an excellent example for the generation of multisolitons, which is interesting in the sense that it can be used to explain the enlarged capacity of transport existing in plasma system. It is relevant to stress that the linear stability analysis provides a very efficient way of probing soliton stability, and particularly its shape invariance under translation.

5. Conclusion

Dusty plasmas are important for the understanding of many phenomena in various space plasma environments. It is also relevant to many laboratory and industrial plasmas applications, such as low-temperature and low-pressure discharges, as well as to the manufacturing of semiconductors. In the present work, we have derived the CKP equation from a dusty pair-ion-electron plasma model. The effects of plasma parameters on the coefficients of this equation were analyzed and the results clearly show that, charged dust particles in electron-ion plasma can be responsible for the appearance of new types of electrostatic waves including multi-solitary waves, depending on whether the dust grains are considered to be static or mobile. The linear stability analysis of the wave was also conducted and accordingly, we conclude that our soliton solutions are stable in the condition that they co-propagate with one of the eigen-vectors mode of the Legendre associated equation of order 2.

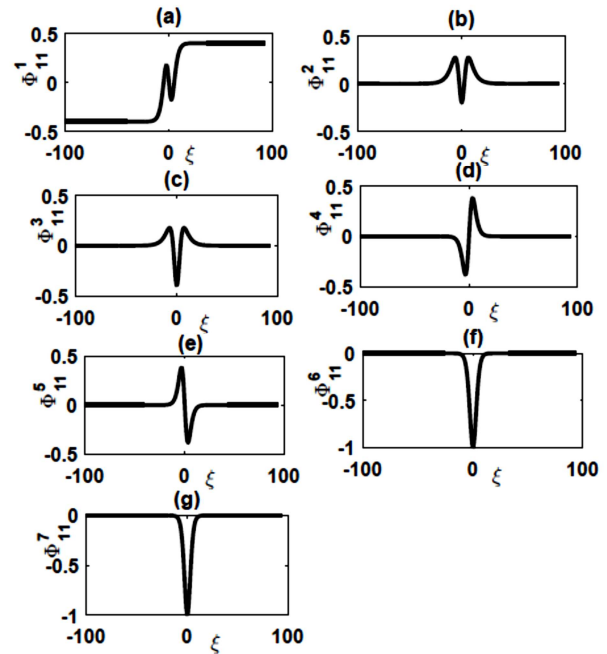


Figure 8. Two-dimensional representation of the wave profiles of seven bound states at time $\tau = 0.01$. The other parameters used are: $a = b = 1$; u

$$= 2; Z_d = 1; \beta = 1; \sigma = 0.01; nd_0 = 0.05n_{-0}; n_{e0} = 0.05n_{-0}; \delta = \frac{n_{-0}}{n_{d0}SZ_d}$$

$$\text{and } \mu = \delta - 1 + \frac{n_{e0}}{n_{d0}SZ_d}.$$

Appendix

Bound State Solution of Legendre Equation

The seven localized modes of the Legendre equation are given by:

$$\Phi_{11}^1 = \Phi_0^1 (\tanh^2(Y - Y_0) - \frac{3}{5}) \tanh(Y - Y_0)$$

$$\Phi_{11}^2 = \Phi_0^2 (\tanh^2(Y - Y_0) - \frac{3}{5}) \operatorname{sech}(Y - Y_0)$$

$$\Phi_{11}^3 = \Phi_0^3 (\tanh^2(Y - Y_0) - \frac{2}{5}) \operatorname{sech}(Y - Y_0)$$

$$\Phi_{11}^4 = \Phi_0^4 \operatorname{sech}^2(Y - Y_0) \tanh(Y - Y_0)$$

$$\Phi_{11}^5 = \Phi_0^5 (\tanh^2(Y - Y_0) - 1) \tanh(Y - Y_0)$$

$$\phi_{11}^6 = \phi_0^6 (\tanh^2(Y - Y_0) - 1) \operatorname{sech}(Y - Y_0)$$

$$\phi_{11}^7 = \Phi_0^7 (\tanh^2(Y - Y_0) - 1) \operatorname{sech}(Y - Y_0)$$

where $\Phi_0^j(k)$ ($j = 1, 2, 3, 4, 5, 6, 7$) are their respective constant amplitudes.

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